Some Operations in the Class Algebra of NBG[1] to Make Different Mathematical Objects

- A philosophical approach -

Marcel S. Bodea
Department of Philosophy
"Babeş-Bolyai" University Cluj-Napoca
bodeamarcel@hotmail.com

Abstract:This philosophical-algebraic presentation aimed at a simple and concise exposition of notions and demonstrations, especially in class algebra. The demonstrations were constructed so that they configure the epistemological objectives on a logical linguistic level, of interest for this analysis: conceptual clarifications and the loading of logical symbols with mathematical signification. My hypothesis is that the propositions of logic have no mathematical content, but become propositions of mathematics, just as, by a different level analogy, propositions of mathematics become propositions of physics. Thus, within the scope of this analysis, a meaningless logical proposition becomes a "mathematical proposition", a proposition with "mathematical signification" by the loading of logical symbols with mathematical significations. As a result, mathematics is regarded here as more than the Wittgensteinian meaning of mathematics as a method of logic.

Keywords: mathematical object, set, class, proper class, membership relation, formula over classes, empty class, universal class, the intersection of the elements of a class, singleton

A referential mathematical object: *ur-elements*

Let us start by indicating that there are mathematical objects important for a conceptual analysis such as this one, which however will not be considered unless comparatively, on a conceptual level, as concepts of reference. One single such object will be mentioned here, important for both classical Cantorian set theory and class algebra: the ur-elements.[2] Two different mathematical objects (and implicitly concepts) will be specified in this context: the set and the ur-element. Intuitive expression: "An *ur-element* is a mathematical object which: has nothing as its component; is not a set; can be a component of a set."[3] Suggestive expression: "Ur-elements are the smallest (most limited) mathematical objects". Intuitive expression: "Comparatively, in class algebra, a

[1] NBG is the abbreviation for "von Neumann-Bernays-Gödel set theory".
[2] Intuitively, they have the meaning of "primary" elements or "atoms".
[3] In the class algebra, the empty class $\varnothing$ is a mathematical object which: has nothing as its component; is a set; can be a component of any class excepting $\varnothing$.

proper class is a mathematical object which: has sets (only sets) as its components; is not a set; cannot be a component of other classes (proper classes or set-classes)”.

Observation: In class algebra, sets are classes which have sets as components, and can be components of other sets as well as proper classes. Suggestive expression: “Proper classes are the largest (most comprehensive, extended) mathematical objects.”[4]

With respect to ur-elements in this limited introductory context, we take the symbol ”$\in$” as intuitively meaning that “something” is an element of “something” which, by definition or construction, is a set: $a \in \underbrace{b}_{\text{set}}$. Referring to the theories operating with the concept of ur-elements, on a conceptual and propositional level,[5] the following remarks are clarifying:

u – *ur-element* ; x – *set* ; y – set

Propositions: ”$u \in x$” ; ”$u \notin x$” ; ”$y \in x$” ; ”$y \notin x$” have sense.

Propositions : ”$x \in u$” ; ”$x \notin u$” ; ”$u \in u$” ; ”$u \notin u$” do not have sense.

Referring to proposition ”$x \in u$”(”$x \notin u$”), anticipating certain later considerations, some clarifications are in order. In the classical (Cantorian) set theory, if e is an element, then proposition $e \in \varnothing$ has sense, but is false. Proposition $e \notin \varnothing$ has sense and is true. Compared to the empty set $\varnothing$ that contains no element and belongs to certain sets,[6] an ur-element contains also no elements, belongs to some sets, but, conceptually, it is fundamentally different from the empty set. In class algebra, propositions ”$x \in y$” ; ”$x \notin y$”, and ”$x \in U$” ; ”$x \notin U$”, where x and y are sets and U a proper class (which is not a set [here the class of all sets][7]) both have sense. (For a proper class P, different from U proposition ”$x \in P_r$” it can be true or false. Proposition ”$x \in U$” is always true, while ”$x \notin U$” is always false.[8]) As long as, by the construction of the theory, *no* proper class

[4] In some sense, the ur-elements are minimal ”mathematical objects” while proper classes are maximal ”mathematical objects”. (Observation: Minimal and maximal are only suggestive formulations, and do not relate to any order relations per se.)

[5] A mathematical proposition is a mathematical statement such as: ”There are infinite sets.”; ” $5 \neq 7$”.

[6] $S = \{1,2\};\ P(S) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\};\ \varnothing \in P(S)$

[7] In the paragraph *Russell’s Paradox*, the class of all sets is noted with Ω, while in class algebra the class noted with U is the universal class. It is shown that U = Ω. The difference in notation only counts where these are explicitly introduced as denoting objects and concepts of a certain type, specific for the given mathematical context. Thus at the level of proper classes no difference is made.

[8] These aspects are important in demonstrating certain theorems in class algebra. (Some with non-intuitive results).

belongs to a class – in the sense that *there cannot be a membership relation* –, propositions "$U \in x$" ; "$U \notin x$" and "$U \in U$" ; "$U \notin U$" are neither true nor false, they simply do not have sense. Thus, in a brief formulation, ur-elements are in this sense dual to proper classes: ur-elements cannot have components that belong to them, proper classes are not components, cannot belong to classes. In turn, as it will be shown, some propositions of class algebra, such as $\cap\varnothing = U$, although intuitively "contradictory", do have a mathematical sense.

With reference to ur-elements and previous clarifications, the logical-symbolic notation:

$$\forall u(\exists x(u \in x)) \wedge \forall u(\forall x(x \notin u))$$

abstracting, in a first approximation, from the legitimacy of using logical quantifiers, has no mathematical sense, although it is logically-formally possible.

The classical set theory

The classical (naïve; Cantorian) set theory has three fundamental concepts: *element*,[9] *set*[10] and *membership relation*[11] between elements and sets. To put it very briefly: in the classical (Cantorian; naïve) set theory[12] a set is a collection of objects (elements) grouped as such (by individualization: $M = \{e_1, e_2, ..., e_n\}$) or grouped by certain properties (the principle of abstraction: $M = \{e | P(e)\}$ shortened symbolic notation for the truth set of the predicate P(e) [in this case a property].[13]

The following observations are made on the level that we consider to be conceptual clarification. Does predicate $P(e)$ individualize only certain objects by their having a certain property or does it also define the set formed by these elements? In the most fundamental, original understanding, predicate $P(e)$ simply individualizes certain objects which this way receive the generic name of *elements*. In this understanding, a

---

9 The *element* is a primary notion with *intuitive content*.

10 So is the notion of *set*.

11 The *membership relation* is also a primary notion with *intuitive content*, but for this relation a *formal algebraic support* will also be constructed. We can also say for $x \in y$: "x belongs to y"; "x is an element of y"; "x is a member of y"; "x is in y".

12 For this theory, the abbreviation CST will be used.

13 $P(x)$ is a predicate variable of valence 1.

notation such as "$e|P(e)$" – as a convention of conceptual clarification – would only mean that "object e" has the property "$P(e)$". To make one step forward in considering "all objects" that have the property "$P(e)$" means to introduce a new concept into the language, even if only as intuitive support: that of "set". And, also, to associate it with a symbol: "$\{e|P(e)\}$", and perhaps also to give an index to it: "$\underbrace{M}_{\text{index}} = \{e|P(e)\}$" . Only this way can it be said that predicate P defines a set. As concerns symbolic writing, $\mathcal{L}$ denotes the language of classical set theory – with the symbol $\{\ \}$ (braces) for representing sets – and $\mathcal{L}^+$ denotes the language obtained from $\mathcal{L}$ by adding a symbol (index) for each set.

After these clarifications, we explicitly state the principle of abstraction for its importance in the present analysis: "Any property (predicate) P(x) *defines*[14] a set, generically noted $\{x|P\{x\}\}$, whose elements are precisely those x elements that have the P(x) property for which proposition P(x) is true".[15] With no further comments, we generalize the previous aspects with the following statement: "The mathematical object constructed (defined) by a *formula* $\varphi(x)$ *over the elements* (whether or not these elements are sets) has the *nature* of a set."[16] This statement is important in the distinction between CST and NBG: in class algebra the mathematical objects constructed by *formulae* $\varphi(x)$ *over classes* can have different *natures*.

Observation: Using as property "the property to be a set", apparently nothing prevents one from speaking about the "set of all sets". In other words, through its premises, the CST does not prevent the construction of sets (mathematical objects) on the basis of certain principles of abstraction formulated highly generally, as is the case of "the set of all sets". This expression leads however to a logical contradiction (the violation of the

[14] The sense in which the property of certain objects *defines* a set and in this sense defines the objects in question as its elements will be clarified later on.

[15] The following equivalence in CST (axiom of extensionality), with direct reference to two sets X and Y: $X = Y \Leftrightarrow \forall x(x \in X \leftrightarrow x \in Y)$ sets the condition for two sets to be equal (in extension): they are equal if and only if they are formed with the same elements. According to the principle of extension, the set $\{x|P\{x\}\}$ is uniquely determined.

[16] There have arised quite soon a series of problems within the CST in connection with the set-like nature of certain objects thus constructed. Let us cite one single example: Cantor's Antinomy: "The set of all cardinal numbers is contradictory".

logical principle of non-contradiction: an object cannot be A and non-A in the same time and under the same relation).

In what follows, the formal algebraic construction of relations will be chosen as subsets of a cartesian product of sets.[17] In this case, with a set of elements E^* and a set of sets M^* given, defined and constructed as agreed above, the binary relation $\in$ algebraically denotes a subset of the Cartesian product $E^* \times M^*$. Let the algebraic definition of the membership relation be a subset of the cartesian product: $E \times M$, where $E = \{e | P(e) = \text{"e is an element"}\}$ is considered the set of all elements and $M = \{S | P(S) = \text{"S is a set"}\}$ is considered the set of all sets. Algebraically: $\in = (E, M, R)$, where $R \subseteq E \times M$. Nothing prevents such a definition in classical set theory with the observation that algebraically the relation is not well-defined if E or M are not sets. In this context the traditional line will be followed, which shows that "the set of all sets" is not a set.[18] For a start, however, E^* and M^* are considered different from E and M.

In the context of conceptual analysis, the following questions can be clarifying: "From the point of view of the membership relation, are the following relations well-defined: (E^*, E^*, R_{E^*}), (M^*, M^*, R_{M^*}), $(M^*, E^*, R_{M^*E^*})$?". The problem goes down to the following questions: "Can an element belong to an element?"; "Can a set belong to a set?"; "Can a set belong to an element?". The first observation has a strictly formal character, with emphasis exclusively on algebraic formalism. If E^* and M^* are accepted as sets, nothing prevents the algebraic construction of relations by the graphs R_{E^*}, R_{M^*}, $R_{M^*E^*}$. In this formal algebraic sense relations are well-defined. There are however restrictions imposed by conceptual limitations.[19] Simple and suggestive examples will be given, compared with corresponding ones of class algebra. A philosophical observation to be made in this case is the following: the symbols of the highly abstract formal algebraic level are loaded with significations, even if all mathematical, and external to

[17] In the class algebra the relations are subsets of a cartesian product of classes.
[18] It is possible to perform a complementary analysis of the concept of "set of all elements". It is not presented here.
[19] The way these "restrictions by conceptual limitations" are correlated with primary notions and the axioms of theories is not treated here. These will only be exemplified with reference to the naïve set theory and the NBG axiomatic theory of sets (class algebra).

this strictly formal algebraic component. In other words, there are different “contents” of significations of the same algebraic formalism in naïve set theory and class algebra.

Let the sets A, B, and C be defined as follows: $A = \{1\},\ B = \{1,2\},\ C = \{1,2,\{1\}\}$

$\{1\} \notin \{1,2\} \wedge \{1\} \subseteq \{1,2\};\ \{1\} \in \{1,2,\{1\}\} \wedge \{1\} \subseteq \{1,2,\{1\}\}$.

Some aspects to note:

$A \in C$. A set can belong to another set. A set can be constructed with any set to which it will belong. Thus, let the predicate that states about a *certain* mathematical object X that it is a set be ”$P(X) = \text{"X is a set"}$”. The notation of the convention of conceptual clarification ”$X | P(X) = \text{"X is a set"}$” only signifies that “object X” has the property of “being a set”. Let us presuppose that we are situated in $\mathcal{L}^{+}$. How is it possible to construct a set which has as an element the set-element X? The emphasis in the answer is on the signification of the conceptual level, keeping the construction however in the formal logical-mathematical framework. In this case the correct formulation of the principle of abstraction is important.[20] Thus, the notation $X | P(X) = \text{"X is a set"}$ is clear in the sense that it states about the individual object X that it has the property of being a set (perhaps among other properties it may have). The notation with braces $\{\ \}$: $\{X | P(X) = \text{"X is a set"}\}$ changes radically (or in this case even “dramatically” for the CST) the signification which comes to involve all objects that have the property of being a set. For now, nothing prevents us from considering the object that this notation implies to be a set. In order to simplify the expression, we index this hypothetic set with Ω: $\Omega = \{X | P(X) = \text{"X is a set"}\}$. What is essential for this case is the fact that we have “squeezed” the individual object X, using only its property of being a set, among the other objects with this property, all members of an object Ω, presupposed to be a set by a principle of abstraction. If it happens to be proved that object Ω is not a set, then the

[20] It is possible to develop a conceptual analysis with its specific relevance for this case, and otherwise, by differentiating two principles of attraction. They will be symbolized thus: I. (the one used in the present analysis) $\{e | P(e)\}$ and II. $\{e \in D | P(e)\}$ where D is a set called domain and $e \in D$ can be considered restriction. As restriction, it can be expressed by $P'(e): e \in D$ and, with this observation, the principle of abstraction is rewritten $\{P'(e) | P(e)\}$, having the form of two principles of abstracion applied to the mathematical objects e.

situation becomes problematic. Let us put this aspect aside, and formulate the requirement to construct another set to which object X belongs by virtue of its property of being a set. Moreover, we investigate the possibility of constructing the set which only has as its element object X by virtue of his property of being a set. In the terminology of a different theory (class algebra), we shall attempt the construction of a "singleton" with *set* X: the *set* with the *single element of set X*.

Let the object X be completely (well) defined as follows: "X is *one* object (only) with properties: $p_1, p_2, ..., p_n, p_{\text{is a set}}$. We note the property of being a set by P(M) and the cumulative property of having the properties $p_1, p_2, ..., p_n$ by $P(p_1, p_2, ..., p_n)$. Let us construct the set of all objects which, if having the property of being sets, then also have the properties $p_1, p_2, ..., p_n$ (and only these). In the symbolization proposed above, it returns to noting the set formed by these objects like this: $\{y | P(M) \wedge P(p_1, p_2, ..., p_n)\}$. According to the logical principle of identity, there is but one such object, which is X. In singleton the expression *set* "**a** *set X*" is corresponding to P(M) and **a** is the correspondent of $P(p_1, p_2, ..., p_n)$. This way, by an object well-defined by its property of being a set but **also** by other properties – this is what "**a** *set x*" signifies – a new set has been constructed, the set formed with element set X: $\{X\}$. So: $\{y | P(M) \wedge P(p_1, p_2, ..., p_n)\} = \{X\}$. In conclusion: "For any set X there is / there can be constructed a set to which it belongs". Moreover, according to the *axiom of extension*, it is immediately apparent that there is the inequality $X \neq \{X\}$ as a result of the two sets being formed with the same elements.[21]

In what follows, some symbolic notations will be presented in order to underline the conceptual differences in CST presented here, without further comments.

Example: $"p_1 : \text{natural number}"$, $"p_2 : \text{even number}"$, $"p_3 : \text{prime number}"$, $"P(p_1, p_2, p_3) = p_1 \wedge p_2 \wedge p_3"$. $X = \{x | P(p_1, p_2, p_3)\} = \{2\}$. Evidently, X (or $\{2\}$) is not associated with property $P(p_1, p_2, p_3)$. In CST, 2 is not a set and therefore it is not associated with property P(M). It can be seen thus that in notation $\{X\} = \{y | P(M) \wedge P(p_1, p_2, ..., p_n)\}$, property P(M) is essential. In the convention adopted

[21] The result can of course be demonstrated formally etc., but this aspect is not important here.

for braces $\{\ \}$, the presence of braces implies the property P(M). How is then set $\{\{2\}\}$ represented? This way: $\{\{2\}\} = \{y | y = \{x | P(p_1, p_2, p_3)\}\}$.

Completion. There are theories which try to define all notions as sets. This is the case of the Zermelo-Fraenkel system. In this system, the natural number 2 is defined as a set like this: $\{\varnothing, \{\varnothing\}\}$. In class algebra NBG singleton is defined like this: $\{x\} = \{y | x \in U \to y = x\}$, where $x \in U$ is the equivalent of the property of being a set (P(M) in the notation above.)[22]

Observations. Any set can belong to another set, and with any set can be constructed another set to which the first belongs. Comparing these with the situation in class algebra, a series of major differences can be noticed. In case of classes such a statement is not true: it is not valid for any class that there is a class which belongs to it. It is only valid for class-sets; what is more, the condition of membership to a class is a necessary requirement for a class to be a set. The conceptual distinctions in the two cases have origins and consequences which significantly distinguish between the theories.[23]

Other remarks are also formulated in this analysis about the sets above.

$A \subseteq B \wedge A \notin B$. A set is included into another set and does not belong to it.

$A \subseteq C \wedge A \in C$. A set can be included into a set and at the same time belong to it.

$1 \in \{1,2\}$ is a legitimate notation and means the membership of element 1 to the set formed with element 1. The general case $X \in \{X\}$ discussed above means the membership of element X (with the property of being a set, among others) to the set formed with element X.

What can be said about notation $1 \in 1$? The conceptual level referred to here (that of CST) implies, at is has been said, three fundamental concepts: *element, set, membership relation*. Regarding the present analysis, limited to a conceptual-philosophical level, we associate the following interpretation of the membership relation $\in$, resulted from certain restrictions imposed by conceptual limitations. The membership relation $\in$ only makes correspondences between elements of different conceptual levels

[22] $U = \Omega$.

[23] Such aspects of class algebra will be treated in the followings, but here things are presented comparatively for the sake of conceptual distinctions and clarifications.

and in only one direction: elements belongs to sets (“element” $\in$ “set”). Formally, only the relation $\in = (E, M, R)$ is legitimate. In particular cases with E^* and M^* relations (E^*, E^*, R_{E^*}), (M^*, M^*, R_{M^*}), $(M^*, E^*, R_{M^*E^*})$ are not legitimate.

Completion. The CST contains notions and operations constructed so that they make correspondences only between elements on the same conceptual level.

Examples. Notation $1 \subseteq \{1,2\}$ is not legitimate. The relation of inclusion is only defined between sets. Notation $1 \cup 2 = \{1,2\}$ is not legitimate because the operation of union is defined only between sets. Furthermore, notation $\{1\} \cap \{2\} = \varnothing$ is legitimate while notation $1 \cap 2 = \varnothing$ is not legitimate.

Answering some of the previously formulated questions, we summarize all these aspects by saying: “An element cannot belong to an element”; “A set taken as a set cannot belong to a set. A set conceptually reconfigured as an element can belong to a set and there is always a set to which it belongs as its element”; “A set cannot belong to an element neither conceptually as a set, nor conceptually as an element”. For the sake of clarity, we illustrate it with some mathematical propositions: $1 \in \{1,2,\{1\}\}$; $1 \in \{1\}$; $1 \notin \{2,\{1\}\}$, in these propositions the conceptual levels “element” and “set” are obvious; the symbol “l” denotes conceptually only the concept of “element”. In propositions $\{1\} \notin \{1,2\} \wedge \{1\} \subseteq \{1,2\}$; $\{1\} \in \{1,2,\{1\}\} \wedge \{1\} \subseteq \{1,2,\{1\}\}$ the conceptual levels are “mixed”: symbol “l” denotes either the concept of “element” or that of “set”.

In the CST, in order for a mathematical entity to be a set conceptually, there must be, with one single exception: $\varnothing$, *elements which belong to it*. The empty set will also be discussed here: there are *no elements that belong to it*.

Continuing the observations above, let us further clarify the concepts of set and class.

The concept of set in its relationship with the concept of class is differently characterized in class algebra: a mathematical entity, especially *a class x is a set if there is a class y, not necessarily a set, to which it belongs*: $\exists y(x \in y)$. As an observation on the level of language, the general expression “y belongs to classes” is not rigorously correct because there are classes that do not belong to classes provided that “belong” implies the relation ”$\in$”. In other words, there are classes which are not sets, proper classes which consist (only) of sets. Furthermore, as in CST the relation $\in$ is not defined (constructed) on *sets* or *elements*, as a binary homogeneous relation in senses $\left(E^*, E^*, R_{E^*}\right)$ and $\left(M^*, M^*, R_{M^*}\right)$, it is **not** defined this way in class algebra either: $\left(C, C, R_C\right)$.

Russell's Paradox. A logical criterion for a conceptual restriction

It has been claimed that, in classical set theory, using as property "the property to be a set", apparently nothing hinders the construction "the set of all sets", and that this construction leads to a logical contradiction by violating the logical principle of non-contradiction. In what follows, emphasis will be placed on the formulation of the mathematical problem, the highlighting of the logical contradiction, and a solution of the paradox, all on a conceptual level.

Formulation of the problem. No premise of CST prevents the construction of the mathematical object "the set of all sets" on the basis of a highly generally formulated principle of abstraction, an object which, according to the same naïve set theory, is a set. Let this be Ω: $\Omega = \left\{X \middle| P(X) = "X \text{ is a set}"\right\}$. According to naïve set theory, in this case the following equivalence can be written: $X \in \Omega \Leftrightarrow X \text{ is a set}$ (E_1), an equivalence which, since Ω was also presupposed as a set, can also be written for Ω: $\Omega \text{ is a set} \Leftrightarrow \Omega \in \Omega$. The principle of abstraction and the role of braces in the mathematical construction $\left\{\ \middle| P\right\}$ preserved at a signification level the concept of set. The membership relation however introduces the conceptual difference: $\underbrace{\Omega}_{\text{element}} \in \underbrace{\Omega}_{\text{set}}$.

Observation. The presupposition that Ω is the set of all sets and Ω is a set, means that Ω belongs to itself, so this means that there are sets that belong to themselves – we do have an example for this. In accordance with the naïve set theory and the logical principle of the excluded middle, if e is an element, and S is a *set*, then there are only two possibilities for the element/set relation, represented by the symbol "$\in$": $e \in S$ or (exclusively) $e \notin S$ (or written differently: $e \notin S \Leftrightarrow \neg(e \in S)$). Moreover, according to the observation above, it creates the possibility of membership or non-membership of a set to itself.

In what follows, we construct the *set* $\mathrm{M}=\{\mathrm{X} | \mathrm{X}\in\Omega\wedge\mathrm{X}\notin\mathrm{X}\}$. The previous considerations allow for the following observations: of $\mathrm{X}\in\Omega$, X is a set and, considering X first as an element and then as a set, we are in one of the following situations regarding the element/set relation: $\mathrm{X}\in\mathrm{X}$ or $\mathrm{X}\notin\mathrm{X}$, in theory neither of the two being interdicted in CST. Consequently, *set* M was legitimately constructed with predicate $\mathrm{P}(\mathrm{X}): \mathrm{X}\notin\mathrm{X}$. Since M is a set, according to the equivalence E_1 it can be written: M is a set $\Leftrightarrow \mathrm{M}\in\Omega$ (E_2).

Highlighting the logical contradiction. Thus, considering M both an element and a set, in the element/set relation there are only two possibilities: $\mathrm{M}\in\mathrm{M}$, and $\mathrm{M}\notin\mathrm{M}$. The logic-mathematical analysis of the two possibilities leads to the following results:

I. It is presupposed $\mathrm{M}\in\mathrm{M}$. Then it results from definition $\mathrm{M}=\{\mathrm{X} | \mathrm{X}\in\Omega\wedge\mathrm{X}\notin\mathrm{X}\}$ and the truth set of predicate $\mathrm{P}(\mathrm{X}): \mathrm{X}\in\Omega\wedge\mathrm{X}\notin\mathrm{X}$ that M is a set and M has the property $\mathrm{M}\notin\mathrm{M}$. Formally: $\mathrm{M}\in\mathrm{M}\Rightarrow\mathrm{M}\notin\mathrm{M}\Leftrightarrow\neg(\mathrm{M}\in\mathrm{M})$ that is, $\mathrm{M}\in\mathrm{M}\Rightarrow\neg(\mathrm{M}\in\mathrm{M})$, contradiction.

II. It is presupposed $\mathrm{M}\notin\mathrm{M}$ ($\neg(\mathrm{M}\in\mathrm{M})$). The legitimate construction of M as a set and the (E_2) yields the result $\mathrm{M}\in\Omega$. This result with the initial presupposition render the logical proposition $\mathrm{M}\in\Omega\wedge\mathrm{M}\notin\mathrm{M}$ true, which means that $\mathrm{M}\in\mathrm{M}$. So a contradiction is obtained in this case as well: $\neg(\mathrm{M}\in\mathrm{M})\Rightarrow\mathrm{M}\in\mathrm{M}$

Ultimately, in order to emphasize the logical content of a contradiction of logical principle, we consider independently the two results $\mathrm{M}\in\mathrm{M}\Rightarrow\neg(\mathrm{M}\in\mathrm{M})$ and $\neg(\mathrm{M}\in\mathrm{M})\Rightarrow\mathrm{M}\in\mathrm{M}$. This way the $\mathrm{M}\in\mathrm{M}\Leftrightarrow\neg(\mathrm{M}\in\mathrm{M})$ equivalence is obtained, which violates the logical principle of non-contradiction.

A solution of the paradox. The direction of solving the paradox treated here, with a philosophical interest for mathematical objects and concepts, is the one offered by class algebra (the von Neumann-Bernays-Gödel version of axiomatic set theory) in interaction with certain aspects of axiomatic set theory in the Zermelo-Fraenkel version.[24]

[24] The interaction is legitimate all the more the von Neumann-Bernays-Gödel axiomatic theory is considered a conservative extension of the Zermelo-Fraenkel axiomatic theory.

Russell's paradox illustrated the way how logic can represent a criterion for the imposition of conceptual restrictions and distinctions in mathematical theory on a language level, and implicitly of objectual differentiations on the level of "ontological" significations. Certain such aspects will be analyzed in what follows.

The class algebra

The *primary notion* of *class* is used. Classes in general are noted with low case letters: x, y, z, … .

Axiom of Extensionality (A.E.) [corresponds to the *Axiom of Extensionality* of Cantorian set theory]

$$x = y \Leftrightarrow \forall z(z \in x \leftrightarrow z \in y)$$

The following result (R1.) is immediately deduced from this axiom: "Any class x is equal with itself: x = x." With the particular emphasis that any set is equal with itself, as well as any proper class (a class which is not a set; for example U/Ω).

A class x is considered a *set* if there is a class y so that $x \in y$. This is a criterion to assess the nature of *set* of a mathematical object. A set can belong to a set or to a proper class; a set is composed only by sets. Proper classes are not sets but are composed only by sets. In the sense of representation $\underbrace{e}_{\text{element}} \in \underbrace{c}_{\text{class}}$, proper classes (generic notation: c_p) belong to no classes, they contain elements, but cannot be elements. In an approach in which the logical apparatus is a necessary condition – mathematics is such an approach – the proposition $c_p \in c$ is false. However, in a different approach it can be considered illegitimate. Philosophy or semiology can be such approaches.

The present philosophical conceptual analysis, for mathematical results and their interpretation within mathematics, will use the perspective of logic as a necessary condition. Thus, in the interpretive version of the proposed analysis of conceptual clarification, the basic idea is to nuance the Wittgensteinian perspective on the relations between types of propositions: meaningful, meaningless and non-sense. In the spirit of the *Tractatus*, a formal logical proposition is meaningless, but it can become meaningful only by loading the symbols with factual meanings (senses). In the context of this analysis, a meaningless logical proposition becomes a "mathematical proposition" –

mathematics is thus much more than a method of logic – a proposition with a "mathematical sense", by loading logical symbols with mathematical meanings ("senses").

Coming back to class algebra, the notation convention for class x is: $M(x)$ if this class is a set. When for certain theoretical interests the relation between class-set elements and proper class has to be emphasized, the $M(x)$ representation is substituted by the representation $x \in U$ having the same meaning, where U is the proper class of all sets.[25]

The formula defined over classes which contains the variable x is noted: $\varphi(x)$.

Observations.

i. Proper classes cannot be connected variables (there are no quantifiers above proper classes: $\nexists c_p$; $\not\forall c_p$).

ii. For the mathematical object constructed (defined) by an *over class* $\varphi(x)$ *formula*, it must be demonstrated for each individual case what the *nature* of the mathematical object is: class-set **or** proper class.[26]

Class Definition Axiom (C.D.A.)

The following mathematical proposition:

$$z \in \{x | \varphi(x)\} \Leftrightarrow M(z) \wedge \varphi(z)$$

will be called: Class Definition Axiom (C.D.A.)

Consequence. It results from the Class Definition Axiom (C.D.A.) and the Axiom of Extensionality (A.E.) that:

$$\forall z\left(z \in \left\{ x | \underbrace{x \in y}_{\varphi(x)} \right\} \leftrightarrow M(z) \wedge \underbrace{z \in y}_{\varphi(z)} \right) \Leftrightarrow \forall z(z \in \{x | x \in y\} \leftrightarrow z \in y) \Leftrightarrow \{x | x \in y\} = y$$

Thus, if y is a class, then:

$$y = \{x | x \in y\}$$

In other words, *a class is characterized (defined) by all sets that belong to it.*

[25] $U = \Omega$

[26] The logical connector **or** in this case is an **or-disjunctive**.

The following remarkable classes are defined:

The empty class: $\varnothing = \{x | x \neq x\}$.

This class is important for several reasons. We present one of these reasons. Russell's paradox has taken to a new mathematical object: the class of all sets U, which is not a set. Beyond CST and U, a problem – both in a philosophical and even *mathematical* sense – arises in the construction of class algebra. What is it that ensures the existence of sets? Can a *set* be "shown" apart from the employment of the concept on this intuitive support of CST?

Let us present for a start an *explicit* solution by *explicitly postulating* the existence of a set: the empty class $\varnothing$ as an axiom. In the language of Zermelo-Fraenkel theory the Empty Set Axiom states that "There is a set for which not set is an element". Formally: $\exists x \forall y \neg (y \in x)$. In the language of class algebra, one of the formulations of the empty set axiom is just as explicit: "The empty class is a set". Formally: $\varnothing = \{x | x \neq x\} \wedge M(x)$.

In what follows, we present a more general version, out of philosophical interest, of the existence of a mathematical object by an axiom under an *implicit* form: "There is at least one class-set.", then later we will identify by a theorem such a particular object. The formal representation of the axiom is the variant: $\exists c (c \in U)$.[27] Being certain this way of the existence of at least one class-set, all of the following mathematical propositions (axioms, definitions, inferences) are legitimate and on their basis such an object will be *identified*: it will be demonstrated that the empty class $\varnothing$ is a set (class-set).[28]

The Axiom of Subsets

$$(x \in U \wedge y \subseteq x) \rightarrow y \in U$$

In words: "For any set (class-set) x there is a set y, so that $z \subseteq x \Rightarrow z \in y$."

This axiom gives sufficient condition for a class to be a set.

Theorem. $\varnothing \in U$.

Proof.

[27] Other symbolic representations are also possible, as is also the articulation with class U (defined by $U = \{x | x = x\}$ with over-class fomula: $\varphi(x)$: "$x = x$" etc.

[28] In another, only suggestive expression we can say that we have an "axiom of existence" (A.E.) of a mathematical object and try to actually find such a (particular) object.

From the definition $\varnothing = \{x | x \neq x\}$ and on the basis of (R1.) the following result (R2.) is demonstrated, by reduction to absurdity: “For any class-set x: $x \notin \varnothing$.” In other terms: “For any class-set x the proposition “$x \in \varnothing$” is false”.

The basic idea is that of the use of condition $v(p) = 0$ of the logical definition of material implication $p \to q$ and the criterion of being a set given by the *Axiom of Subsets* (A.S):

x, y– classes; $x \subseteq y \overset{\text{def}}{\Leftrightarrow} \forall z(z \in x \to z \in y)$; A.E.: $\exists c(c \in U)$;

$\varnothing \subseteq c \Leftrightarrow \forall z \underbrace{\left(\underbrace{z \in \varnothing}_{0} \to z \in c \right)}_{1}$;

A.S.: $\varnothing \subseteq c \Rightarrow \varnothing \in y$[29]; $\varnothing \in y \Leftrightarrow \varnothing \in U$.

In conclusion, the over-class formula $\varphi(x) = "x \neq x"$ constructs (defines) a class: the *empty class* $\varnothing$ which is a set (it can be written $M(\varnothing)$).

Observations.

i.) As it will be proven, $\varnothing \in U$ makes possible the construction of new class-set mathematical objects. For example: $\{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \ldots$ [30]

ii.) Leaving aside $\varnothing \in U$ as a theorem, in class algebra the *Axiom of Infinity* postulates the existence of a class-set, the set-nature of the empty class $\varnothing$ in relation with this class-set[31] and the possibility to construct new sets starting from sets which belong to the postulated class-set, with the important property that these new sets also belong to the same class-set:

The Axiom of Infinity[32]

There is a set x with the properties:

---

[29] Axioma submulțimilor nu spune nimic despre clasa-mulțime y, în particular nu interzice ca aceasta să fie c. Exemplul anterior din TCM: $A = \{1\}$, $C = \{1, 2, \{1\}\}$, $A \subseteq C$, $A \in C$, este sugestiv în acest sens. Mai mult, nici nu ar fi necesar ca y să fie mulțime. Toate aceste precizări nu sunt însă necesare pentru demonstrație.

[30] The signification for $\{\varnothing\}$ and $\{z\}$ with z–class will be given later on.

[31] This result in the above theorem would be directly obtained considering that $y = c$ by virtue of the fact that the existence of the class-set c was postulated by the axiom of existence: thus there is a set c so that $\varnothing \in c$.

[32] In a different formalization: $\exists x(\varnothing \in x \wedge \forall y(y \cup \{y\} \in x))$

1.) $\varnothing \in x$

2.) $y \in x \Rightarrow y \cup \{y\} \in x$

Universal Class $\quad U = \{x | x = x\}$.

We demonstrate that this class (U) is the class ("set") of all sets (Ω) in the formulation of Russell's Paradox.

Theorem. $\quad U = \Omega$

Proof.

This demonstration is based on the definition of the concept of set in its association with the membership relation to a class. It follows a series of logical steps which outline conceptual aspects deriving explicitly from the "content" of the steps and not the accompanying comments.[33] We take mathematical object "the class of all sets" and note it with Ω. According to the notation convention M(x) for x class-set, this goes back to the representation: $x \in \Omega \leftrightarrow M(x)$ or: $\Omega = \{x | M(x)\}$, or, in the representation in class-form according to *Class Definition Axiom* (C.D.A): $\Omega = \{x | x \in \Omega\}$. From the Axiom of Extensionality (A.E.) and the definition of inclusion "$\subseteq$" results the following equivalence for classes: $x = y \Leftrightarrow x \subseteq y \wedge y \subseteq x$. In this case: $U = \Omega \Leftrightarrow U \subseteq \Omega \wedge \Omega \subseteq U$.

"$U \subseteq \Omega$"

$$z \in U \to z \in \left\{ x | \underbrace{x = x}_{\varphi(x)} \right\} \leftrightarrow M(z) \wedge \varphi(z) \leftrightarrow M(z) \wedge \underbrace{(z = z)}_{R1.} \leftrightarrow M(z) \leftrightarrow z \in \Omega$$

so $\quad \forall z(z \in U \to z \in \Omega) \Leftrightarrow U \subseteq \Omega$.

"$\Omega \subseteq U$"

$$z \in \Omega \leftrightarrow M(z) \leftrightarrow M(z) \wedge \underbrace{\overbrace{z = z}^{R1.}}_{\varphi(z)} \leftrightarrow M(z) \wedge \varphi(z) \to z \in \{x | \varphi(x)\} \leftrightarrow z \in \{x | x = x\} \leftrightarrow z \in U$$

so $\quad \forall z(z \in \Omega \to z \in U) \Leftrightarrow \Omega \subseteq U$

consequently $U \subseteq \Omega \wedge \Omega \subseteq U \Leftrightarrow U = \Omega$ the two mathematical objects: *universal class* and the *class of all sets* are one and the same object, they represent the same class.

[33] From a strictly formal point of view, the demonstration can be schematized and the number of steps reduced.

This class is not a set, it is another mathematical object (it CANNOT be written $M(U)$ or $M(\Omega)$). It is a class called: *proper class*.[34] Important observation: From a *logical-linguistic* viewpoint, presupposing that this "mathematical object" Ω is a *set* in the classical sense of Cantorian set theory, one gets to a logical-mathematical contradiction. This contradiction was mathematically eliminated by the construction of axiomatic systems of set theory (Zermelo-Fraenkel; Von Neumann-Bernays-Gödel [class algebra] etc.). In class algebra U/Ω is a class which is not a set, any set is a class, and any class is only characterized by the sets that belong to it.

For two classes x and y *the intersection* "$\cap$" *of the classes x and y* is defined as the class:

$$x \cap y = \{z | z \in x \wedge z \in y\}$$

This definition is also found in classical set theory. It generates a class (a mathematical object) starting from the explicit consideration of one single class.

For the x class the following class is defined:

The intersection of the elements of the x class[35]

$$\cap x = \{z | \forall y (y \in x \rightarrow z \in y)\}$$

Expressed in words, $\cap x$ is the class formed of the sets that belong to all the sets that belong to the x class. This class and its definition is specific to class algebra. It has an intuitive support in the classical definition of set intersection in Cantorian theory applied to the "composition" of class x.

In the demonstrations that follow, the logical definition of material implication is essential.

$$x = \{y | M(y) \wedge P(y)\} \text{ or inferring that y is a set } x = \{y | P(y)\}$$

which is a short notation for the truth set of predicate P(y). It is important to emphasize that it is the consideration in a global sense of the "truth" of the *compound proposition*

[34] Observation. There are other theories which accept the existence of proper classes among fundamental mathematical objects, or, at a conceptual level, the concept of proper class as a fundamental concept. The Morse-Kelley version of set theory admits the existence of proper classes as fundamental mathematical objects; moreover, it allows by its axioms the quantification over proper classes, which is not permitted in NBG (as quantification is limited to sets only).

[35] Another name for this "operation" is: Set Product (For a class, all sets that are elements of each set that belongs to the class are members of the Set Product.)

after the sign ”|”, regardless of the possible truth values of the elementary proposition. The following is a suggestive example in this sense, written for the sake of simplicity in the version of naïve set theory, and also for the sake of simplicity considering only natural numbers:

$$X = \left\{ y \middle| \underbrace{\underbrace{0 \le y \le 4}_{p} \vee \underbrace{2 \le y \le 5}_{q}}_{P = p \vee q} \right\} = \{0,1,2,3,4,5\}$$

Thus, for y = 1 the proposition p is true while the proposition q is false, for y = 5 q is true and p is false, for y = 3 both p and q are true, and for y = 7 both p and q are false. However, what matters is only the truth of the compound proposition P (after the sign ”|”) that we have conventionally represented here as “$P = p \vee q$”. In what follows, we take this observation in a general sense and avoid logical details in notations and representations. Further on, two mathematical results are presented in the form of theorems.

Theorem. (of the intersection of an empty class with itself) $\varnothing \cap \varnothing = \varnothing$

Proof.

According to the definition of the intersection of two classes x and y: $x \cap y = \{z | z \in x \wedge z \in y\}$ we have $\varnothing \cap \varnothing = \{z | z \in \varnothing \wedge z \in \varnothing\}$ from where, applying the idempotent law, the logical operator “$\wedge$” is obtained: $\varnothing \cap \varnothing = \{z | z \in \varnothing\}$. Next, we analyze class $\{z | z \in \varnothing\}$. According to the definition of the empty class and the *Axiom of Classification*, we have: $z \in \varnothing \Leftrightarrow z \in \{x | x \neq x\} \Leftrightarrow M(z) \wedge z \neq z$. According to the result (R1) the proposition $z \neq z$ is false and so $M(z) \wedge z \neq z \Leftrightarrow z \neq z$. It results then: $z \in \varnothing \Leftrightarrow z \in \{x | x \neq x\} \Leftrightarrow M(z) \wedge z \neq z \Leftrightarrow z \neq z$. Substituting this result in the equality of $\varnothing \cap \varnothing = \{z | z \in \varnothing\}$, we obtain: $\varnothing \cap \varnothing = \{z | z \in \varnothing\} = \{z | z \neq z\} = \varnothing$.

This result presents no logical contradiction. The result is intuitive.

The basic idea in the following demonstration is the use of the condition $v(p)=0$ of the logical definition of the material implication $p \to q$, and also the criterion of being a set offered by the *Axiom of Subsets* (A.S.).[36]

Theorem. (of the intersection of the elements of an empty class) $\cap\varnothing = U$.

Proof.

For empty class $\varnothing$ we explicitly write *the intersection of the elements of class* $\varnothing$ according to definition: $\cap\varnothing = \left\{ z \middle| \forall y \left(\underbrace{y \in \varnothing}_{p} \to \underbrace{z \in y}_{q} \right) \right\}$. Therefore only that y and z are presupposed for which we have sets M(y) and M(z), and according to (R2) the proposition p is always false. Then for any set y [$(\forall y)$] the implication $p \to q$ is true (the particular result mentioned above). In other words, the compound proposition after "|": $\forall y(y \in \varnothing \to z \in y)$[37] is always true. In conclusion, since we have presupposed the description of classes only by sets, it results that in these conditions z can represent any set and therefore class $\cap\varnothing$ is the class of all sets, that is, universal class U, and thus the theorem $\cap\varnothing$ = U is demonstrated.

This result presents no logical contradiction. The result is non-intuitive. It is just non-intuitive. Logically and mathematically it is correct. Basically it is the consequence of results based on logic using mathematically significant symbols. The propositions of logic are purely formal, logical symbols mean nothing outside logic.[38]

We formulate the observation in this context that there might be problems of interpretation by the "loading" of logical symbols with certain significations taken over from a language for which logic may or may not be a necessary condition.[39] Here we deal with the loading of logical symbols with mathematical significations. In the field of fact-based sciences we deal with the loading of mathematical symbols (mathematical formalism) with empirical significations (for instance physical: mechanics,

[36] Another symbolic version of the *Axiom of Subsets*: $(M(x) \wedge y \subseteq x) \to M(y)$.

[37] In logical details there is a compound logical proposition $P(y,z)$ with bound variable y and free variable z etc., aspects which are not of interest here.

[38] "The propositions of logic therefore say nothing. (They are the analytical propositions.)" - Tractatus 6.11

[39] For instance, for mathematical language and scientific languages logic is a necessary condition, but for artistic or theological languages logic is not (necessarily) a necessary condition.

electrodynamics, etc.). In the field of philosophy logic, mathematics, theoretical physics is loaded with epistemological significations (metaphysical, theological, etc.). For example, Wittgenstein explicitly states that mathematics is a method of logic and the propositions of mathematics are the equations, that it, meaningless propositions (pseudo-propositions).[40] Even if we admit this statement, it must be emphasized that the propositions of logic do not have mathematical content. On this line of interpretation mathematics is, among others, the loading of logical propositions with mathematical content.[41] Moreover, mathematical proposition also express nothing about facts.[42] Starting with Galilei, the *Book of Nature* is written in a mathematical language and is usually shown by equations.[43] It must be said again that the propositions of mathematics do not have a factual, metaphysical, theological, artistic or other content.[44]

These aspects regarding the loading of logical symbols with mathematical significations are completed with an observation about the demonstrations using the definition of the logical implication with false antecedent ($\underbrace{p}_{0} \to q$). The observation refers to the *vacuous truth* associated to some propositions which claim about the elements of an empty set to have a certain property.[45] There has been explicated the first

---

[40] "Mathematics is a logical method. The propositions of mathematics are equations, and therefore pseudo-propositions." - Tractatus P 6.2.

[41] In a semiologically less accurate, yet more "relaxed" and therefore more suggestive expression, mathematics is the loading of logical propositions with mathematical contents, significations, senses, etc.

[42] "Mathematical propositions express no thoughts." - Tractatus P 6. 21. and " In life it is never a mathematical proposition which we need, but we use mathematical propositions only in order to infer from propositions which do not belong to mathematics to others which equally do not belong to mathematics. (In philosophy the question "Why do we really use that word, that proposition?" constantly leads to valuable results.)" – Tractatus P 6.211.

[43] "The logic of the world which the propositions of logic show in tautologies, mathematics shows in equations." - Tractatus P 6.22.

[44] "Theories which make a proposition of logic appear substantial are always false. One could e.g. believe that the words "true" and "false" signify two properties among other properties, and then it would appear as a remarkable fact that every proposition possesses one of these properties. This now by no means appears self-evident, no more so than the proposition "All roses are either yellow or red" would sound even if it were true. Indeed our proposition now gets quite the character of a proposition of natural science and this is a certain symptom of its being falsely understood." - Tractatus 6.111. We take these considerations as valid also in the case of mathematical propositions.

[45] A statement S is "vacuously true" if it resembles the statement $P \Rightarrow Q$, where P is known to be false. Statements that can be reduced (with suitable transformations) to this basic form include the following:

$\forall x : P(x) \Rightarrow Q(x)$, where it is the case that $\forall x : \neg P(x)$.

$\forall x \in A : Q(x)$, where the set A is empty.

$\forall \xi : Q(\xi)$, where the simbol ξ is restricted to a type that has no representatives.

step of loading a logical proposition $p_{(logica)}$ with mathematical signification: $p_{(logica)} \mapsto p_{(matematica)}$[46], where the mathematical proposition $p_{(matematica)}$ is "$x \in \varnothing$". This proposition is *mathematically false*. The next step is the consideration of the (compound) logical proposition "$p \rightarrow q$". The next step, starting from the mathematical falsity of the mathematical proposition "$x \in \varnothing$" is the attribution of the value of logical falsity to the proposition "p" of the logical proposition "$p \rightarrow q$". The next step is the acceptance by logical considerations of the affirmation that "proposition p is false, the implication (the compound proposition "$p \rightarrow q$") is true", having its consequences on the logical truth value of the proposition "q": it can be true or false. The next step is the consideration of a hybrid logical-mathematical proposition: $x \in \varnothing \rightarrow q$, where the first proposition is mathematical, while the second is for now considered purely logical. The next step is the loading of the mathematical proposition with a signification which is *alien to mathematics*: "the elements of an empty set have a certain property". Mathematically, the signification of the proposition "$x \in \varnothing$" is the negation of the proposition "$x \in \varnothing = \{x | x \neq x\}$".[47] The next step in the philosophical interpretation would be the introduction of *vacuous truth* in mathematics and the analysis of its consequences on linguistic level, ontological level, etc. At this point we remain at this illustration of possible further interpretations.

For more clarity let the following short completions be accompanied by a simple illustration. The propositions of logic and mathematics have their own "formal criterion" of truth which must be ensured. These are necessary conditions for the rational description of the world of facts. However, the propositions of logic and mathematics are not also sufficient conditions for the description of the world of facts. The propositions about facts must meet other specific criteria of truth as well. Scientific propositions must

---

Vacuous truth is usually applied in classical logic, which in particular is two-valued. However, vacuous truth also appears in, for example, intuitionistic logic in the same situations given above. Indeed, the first two forms above will yield vacuous truth in any logic that uses material conditional, but there are other logics which do not." - http://en.wikipedia.org/wiki/Vacuous_truth

[46] The symbol "$\mapsto$" denotes this load, by associating a mathematical proposition with a logical proposition which ultimately comes back to replacing the logical proposition p with the mathematical proposition p_m.

[47] The possible consequences on the level of logical predicates with mathematical interpretations are not analyzed here.

meet the criteria of an “exterior truth” of empirical correspondence with facts.[48] For example, a mathematical proposition is formal, symbolical, having no factual meaning, but by “loading” mathematical symbols with empirical signification (physical in this case) the proposition acquires a factual sense and only thus can it speak about facts. In this sense, a physical proposition must be mathematically true and empirically true.

As a completion, we construct, as merely a philosophical “application”, and without discussing the legitimacy or illegitimacy of the procedure, the loading of mathematical symbols with metaphysical content and offer a possible reading of this “mathematical signification of mathematical results”. The philosophical relevance or lack of relevance of this endeavour is not discussed here, it is just an illustration.

Thus: $\varnothing \mapsto \text{nothing}(\text{nothingness})$

$U \mapsto \text{universals}(\text{all things})$

The metaphysical reading of mathematical theorems:

$\varnothing \cap \varnothing = \varnothing$: The intersection of nothing is nothing!

$\cap \varnothing = U$: The intersection of nothing is all things!

The need for conceptual restrictions

In order to highlight the problem, it must be specified what is the signification of the description of a class by the *principle of abstraction*, *adapted* to class algebra, understood here as starting from the logical signification of the notion of predicate. When the notion of logical predicate is introduced in logic, it is defined as:

“Let X be a *set* and $n \in N^*$. It is called a n-ary logical predicate on *set* X any function $P : X^n \to \boldsymbol{\beta}$, where $X^n = \underbrace{X \times X \times ... \times X}_{\text{de n ori}} = \{x = (x_1, x_2, ..., x_n) | x_i \in X, i = \overline{1,n}\}$ is the Cartesian product of *set* X with itself for n times, and $\boldsymbol{\beta}$ is the *set* of all logical propositions.” The highlights are meant to emphasize the concept of *set*. Under these circumstances, to what extent can one speak about logical predicates or logical formulas defined by classes? For instance, in the formulation of the Class Definition Axiom

[48] ”It is the characteristic mark of logical propositions that one can perceive in the symbol alone that they are true; and this fact contains in itself the whole philosophy of logic. And so also it is one of the most important facts that the truth or falsehood of non-logical propositions can not be recognized from the propositions alone.” - Tractatus P 6.113

($z \in \{x|\varphi(x)\} \Leftrightarrow M(z) \wedge \varphi(z)$) of class algebra, the notations $M(x)$ to denote that class x is a set, and $\varphi(x)$ to denote a formula defined by classes, containing the variable x-class are introduced beforehand.[49]

In order to underline the importance of the conceptual level in various mathematical approaches, the following example is presented with respect to Zermelo-Fraenkel theory.[50] The basic idea of the theory, formulated here with reference to conceptual analysis, is to define all notions as sets. In this sense all mathematical entities of predicates and formulas (the variables) are sets. Next, we consider two of the axioms of this theory: the Axiom of Separation and the Axiom of Replacement.

The Axiom of Separation

The notion of *definable part*. Let the language be $\mathscr{L}^+$ (as previously defined) and let $\varphi(x)$ be a formula of this language. We call *definable part* of D the **collection** of all elements x from D so that $\varphi(x)$ is true. With this notion the Axiom of Separation is explained this way: “If D is a **set**, then any definable part of D is also a **set**.” Formally, with the symbols and significations of $\mathscr{L}^+$, it can be written: $\{x \in D|\varphi(x)\}$.

The notation $\{x \in D|\varphi(x)\}$ in the context of the formulation of the Axiom of Separation needs conceptual clarifications. For the sake of clarity and convenience, notions of class algebra will also be used in the explications.[51] In the formula $\varphi(x)$, x covers the collection (*class*) of all sets which is not a set. In this case $\{x \in D|\varphi(x)\}$ can be rewritten: $\{x \in U|\varphi(x)\}$, where U is a *proper class*. It must be specified that there is not one single proper class (U/Ω), but there are different proper classes. Generally speaking, different collections of “totality of mathematical objects of a certain type (especially with a certain property)” form proper classes. For instance: the class of all sets; the class of all cardinal numbers; the class of all ordinal numbers; the class of all groups, etc. In these circumstances it is natural to presuppose that a subclass of U, formed of course of sets,

[49] For example, Breaz Simion, Covaci Rodica - *Elemente de logică, teoria mulțimilor și aritmetică* (Elements of logic, set theory and arithmetics), Editura Fundației pentru Studii Europene, Cluj-Napoca, 2006, p. 152.
[50] The Zermelo-Fraenkel theory will be abbreviated to ZF.
[51] Some notions are defined in this text, while others are not. Basically, the main notions and results of class algebra are supposed to be known.

can be a proper class. For example, $\{x \in U | P(x) = \text{"x is group"}\}$ represents a collection of sets which is the class of all groups. Coming back to the notation by formula $\varphi(x)$, $\{x \in U | \varphi(x)\}$, theoretically formula φ could select from the class of all sets a subclass of sets so that this subclass would be a proper class.

The Axiom of Replacement

The notion of *application*. If $\varphi(x, y)$ is a formula of $\mathcal{L}^+$, then we can say that φ **defines** an **application** if for any x there is one single y, so that $\varphi(x, y)$ is true. With this notion the Axiom of Replacement is explained this way: "Any *definable application* whose domain is a set is a function." In the language of the *Axiom of Separation* the *Axiom of Replacement* is explained: "If D is a set then the *restriction of the application* **defined** by φ at D is a function." Formally, with the symbols and significations of $\mathcal{L}^+$, and with the symbol (x, y) which signifies an ordered pair,[52] it can be written:

$$\{(x, y) | x \in D \wedge \varphi(x, y)\}.$$

Remark. In what regards the correlation formula/function, the Axiom of Replacement offers a condition for a formula which "defines" an application to be a function: "If a formula $\varphi(x, y)$ defines an application then it can be represented as a function f *with the condition* that φ is restricted to a domain D which is a set."

The following is an illustration of the signification of the Axiom of Replacement from the perspective of conceptual distinctions. In order to introduce a new axiom, the Axiom of Infinity, and to construct then the set of natural numbers (natural numbers are mathematical objects of utmost importance (!)), ZF theory introduces a new notion: the successor application. Thus for any *set* x it is defined:[53]

$$S(x) = x \cup \{x\}$$

[52] $(x, y) = \{\{x\}, \{x, y\}\}$

[53] By convention, in this theory sets are usually noted with low case letters.

As an immediate result: $S(x)$ is a set (if x is a set then $\{x\}$ is also a set[54] and the union of two sets is a set[55]).

The next important step is the *addition* of $S(x)$, understood here as a "$\varphi(x)$ formula" at $\mathcal{L}^+$ language. In this case $\varphi(x, S(x))$ at $\mathcal{L}^+$ language.

The successor application $S(x) = x \cup \{x\}$ as application is not a function since its definition domain D is the set of all sets which is not a set. The successor application is however definable if its domain is restricted to a set. In the language of class algebra this is a conceptual restriction from the concept (/object) of proper class to the concept (/object) of class-set. (From the point of view of mathematical objects the objectual reference is changed).

The construction of mathematical objects in class algebra

To conclude, with respect to mathematical objects and the proposed conceptual analysis, we consider the following construction possibilities representative, sets and proper classes, in class algebra. These are presented and adapted so that their conceptual significations are emphasized.

Definition. *Singleton*

$\{x\} = \{y | x \in U \to y = x\}$

Theorem. $x \notin U \Leftrightarrow \{x\} = U$ ($\neg M(x) \Leftrightarrow \{x\} = U$)

Proof.

(The demonstration of the theorem, highlighting conceptual aspects, avoids the simpler ways of construction based on different results [such as: $x \in U \Rightarrow \{x\} \in U$; $y \in \{x\} \Leftrightarrow y = x$] and grounds such results on the theorem).

"$\Rightarrow$" ($x \notin U \Rightarrow \{x\} = U$) In the definition of the singleton, from $x \in U \to y = x$, for the condition $x \notin U$ the proposition $x \in U$ is false ($\underbrace{x \in U}_{0}$) so the implication is true

[54] For CST we have shown that if x is a set then $\{x\}$ is a set. A similar result is demonstrated about class algebra. In this context of ZF theory we shall accept without demonstration the theorem that if x is a set then $\{x\}$ is a set.

[55] This is also a theorem of ZF theory.

($\underbrace{\underbrace{x \in U}_{0} \to y = x}_{1}$). According to the consequence of the axiom of classification, an class is defined only by sets, consequently $\underbrace{M(y)}_{1}$ ($y \in U$), so $\underbrace{M(y) \wedge (x \in U \to y = x)}_{1}$ is still true. In the definition of the singleton, writing everything explicitly, we have: $\{y | M(y) \wedge (x \in U \to y = x)\}$, which means exactly the class of all sets. So $x \notin U \Rightarrow \{x\} = U$.

"$\Leftarrow$" ($\{x\} = U \Rightarrow x \notin U$) Let us presuppose $\{x\} = U$. $\{x\} = U \leftrightarrow \{x\} = \{y | y \in U\}$. So $(\{x\} = U \wedge \{x\} = \{y | x \in U \to y = x\}) \Leftrightarrow (\{x\} = \{y | y \in U\} \wedge \{x\} = \{y | x \in U \to y = x\})$. From where results: $\underbrace{\{y | y \in U\}}_{\text{I.}} = \underbrace{\{y | x \in U \to y = x\}}_{\text{II.}}$. In I. y denotes any set. We presuppose that the x from the singleton is a set ($x \in U$). Then it represents a certain set, a unique set. According to I. and the consequent of the implication of II. it results that a certain set is equal with any set which, logically speaking, violates the law of identity. As a result, the presupposition $x \in U$ in the context of singleton $\{x\}$ is false ($\underbrace{x \in U}_{0}$), so we have $\neg(x \in U) \leftrightarrow x \notin U$, that is, $\{x\} = U \Rightarrow x \notin U$, and thus the theorem $x \notin U \Leftrightarrow \{x\} = U$ is demonstrated.

This theorem allows for reaching a direct result next, also relevant for conceptual analysis.

Theorem. $\{U\} = U$

The theorem $x \notin U \Leftrightarrow \{x\} = U$ says that if "something" is not a set, that is, that "something" is a proper class, then the singleton of that proper class is a universal proper class. It has been specified however that there are distinct proper classes. This way the singleton, besides preserving the conceptual nature of proper class of the new mathematical object, reduces the extension of "proper classes" objects to only one: universal class, and in this sense it brings different proper classes "to a common conceptual denominator". In a metaphysical interpretation, without dwelling too much on it, the singleton which is in a way the exterior individualization of a proper class, recovers for this new proper class the sets which are absent from its content.

Observation. $x \neq U \Rightarrow x \neq \{x\} = U$. Metaphysical interpretive games can be interesting, but our only concern here is an analytic interest of conceptual clarification. Adding also the result $\{U\} = U$, it can be stated that by the singleton the construction of proper classes "is saturated" in at most one step. Let us agree only here and only for this analysis to call the theorem $x \notin U \Leftrightarrow \{x\} = U$ the *theorem of objectual limitation*. Things are not the same for class-sets.

Extending those said above, the "conceptual signification" of the following theorem is also of interest for the analysis. It allows the construction of a new mathematical object of the same nature, characterized exclusively by the object that it starts from, namely a mathematical object of a particular nature: a set. The construction is permanently open, generative. Conceptually, it is the same level: the concept of set, the basic (original) objectual reference always remains the same, but the new possible objects are always different. Using, for the sake of suggestiveness alone, an analogy of a very different field, biology, it would be a sort of "cellular division".[56] ("set x" and "set $\{x\}$", $x \neq \{x\}$; $\{x, \{x\}\}$).[57]

Theorem. $x \in U \Leftrightarrow \{x\} \in U$

Proof.

I. . "$\Leftarrow$" ($\{x\} \in U \Rightarrow x \in U$) In order to stay, if possible, in the objectual-conceptual field outlined by the previous interpretation, we attempt the construction of a demonstration on the basis of the *theorem of objectual limitation*. We presuppose the proposition $\{x\} \in U$ to be true. We use the theorem $x \notin U \Leftrightarrow \{x\} = U$. If $x \notin U$ then $x \notin U \to \{x\} = U$ but $U \notin U$ and from there it results that: $(\{x\} = U) \wedge (U \notin U) \to \{x\} \notin U$, an absurd result which contradicts the hypothesis: to presuppose $\{x\} \in U$. Thus the implication $\{x\} \in U \Rightarrow x \in U$ is demonstrated.

---

[56] The analogy was presented only for its plasticity.

[57] The "preserved" conceptual level is general: that of the concept of set. In a different respect, the new mathematical objects can get very different significations which, intuitively, on the level of distinctions, can be familiar. Thus, let us present very briefly the following particular example: $\varnothing = 0; \{\varnothing\} = 1; \{\varnothing, \{\varnothing\}\} = 2; \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = 3$ etc., with the remark that the construction of these new mathematical objects takes something more than the result of the theorem ("division"), it takes a supplementary mathematical context (a new "mathematical environment" in which natural numbers (!) may "evolve"). The game of analogies is limited to this much.

II. ”$\Rightarrow$” ($x \in U \Rightarrow \{x\} \in U$)

i.) In what follows, we attempt to construct a demonstration on the basis of the *theorem of objectual limitation* $x \notin U \Leftrightarrow \{x\} = U$. The hypothesis is $x \in U$. We presuppose $\{x\} \notin U$. Then $\{x\} \notin U \leftrightarrow \{\{x\}\} = U$. In this first context we only have proper class U. The criterion which makes U proper was considered here “that of the logical contradiction” to which it reaches (Russell’s paradox). If the only proper class *would be* U, then $(\{x\} \notin U \leftrightarrow \{\{x\}\} = U) \Rightarrow \{x\} = U \Rightarrow x \notin U$, which contradicts the hypothesis and the theorem is demonstrated. However, the concept of proper class is not associated with a single representative. From $U = \left\{ x \middle| \underbrace{x = x}_{\varphi(x)} \right\}$ it shows that a formula $\varphi(x)$ can construct a proper class. Thus, in principle, a formula $\varphi(x)$ over class U can construct a proper class different from U. (Any proper class, as long as it is formed of sets, is included in U). Exemplifying by the predicative construction (with $P(x)$), class $\{x \in U | P(x) = "x \text{ is group}"\}$ is a proper class. In these conditions, when there are other proper classes in addition to U, it can no longer be concluded that $(\{x\} \notin U \leftrightarrow \{\{x\}\} = U) \Rightarrow \{x\} = U$ because $\{x\} \notin U$ may signify for $\{x\}$ any proper class and thus there is no ultimate step of the demonstration $\{x\} = U \Rightarrow x \notin U$, that is, the theorem is not demonstrated, so we cannot speak about the possibility of conceptual construction in the given sense.

ii.) To demonstrate the implication $x \in U \Rightarrow \{x\} \in U$, we may use, for example, the axiom of limitation of size formulated in a synthetic way: “O class C is proper if and only if there is a bijection of class between class C and the universal class U (the class of all sets).” (In another short formulation, the axiom of limitation is: $x \notin U \Leftrightarrow |x| = |U|$, where U is the class of all sets.) It can be observed that $\{x\} \notin U \leftrightarrow \{x\} \sim U$.[58] From the initial presupposition $x \in U$ and the definition of the singleton $\{x\} = \{y | x \in U \to y = x\}$ results that class $\{x\}$ has exactly one element: set x, which is contradicted by $\{x\} \sim U$. This way the implication $\{x\} \in U \Rightarrow x \in U$ is also demonstrated. In conclusion, theorem $x \in U \Leftrightarrow \{x\} \in U$ is demonstrated.

[58] The equipotency relation “~” between classes is implied.

This demonstration is not interesting however on a conceptual level. Consequently, we configure another demonstration.

iii.) The *Axiom of Substitution*. If $f : x \to y$ is a function and x is a set, then $\{f(u) | u \in x\}$ is a set.

We build up a demonstration using the *Axiom of Substitution*. According to the definition of the singleton, *set* x can form *class* $\{x\}$. The class is correctly constructed since it is formed of sets. We consider $y = \{x\}$. We legitimately[59] build the surjective ("total surjective") function $f : x \to \underbrace{\{x\}}_{\text{clasa } y}, u \in x \wedge f(u) \underbrace{\mapsto}_{(=)} x$. Then, according to the Axiom of Substitution, class $\{f(u) | u \in x\} = \{x\}$ is a set. Therefore the theorem $x \in U \Leftrightarrow \{x\} \in U$ is demonstrated.

Observation. The *Axiom of Extensionality* (A.E.) immediately yields the result $x \neq \{x\}$.

These results (the theorem and the observation) allow, on the one hand, for the construction of *new sets* ($x \in U \Rightarrow \{x\} \in U$) starting from sets, and, on the other hand, in certain cases, for the "recognition" of another mathematical object as being a set, starting from a set ($\{x\} \in U \Rightarrow x \in U$).

Although interesting and offering further clarifications, the analysis of the "conceptual" role of braces $\{\ \}$ on "pair classes" will not be performed here. We only mention the definition and two results, with the observation that braces may contain two mathematical objects of different natures, in the sense that, although both are classes, one can be a class-set, and the other a proper class.

Definition. *Pair class*

$\{x, y\} = \{x\} \cup \{y\}$

We shall present three results, with no demonstration or remarks, corresponding to those connected to the definition of the singleton, as two theorems:

Theorem. $\{x, y\} = U \Leftrightarrow x \notin U \vee y \notin U$.

Theorem. $x \in U \wedge y \in U \Leftrightarrow \{x, y\} \in U$.

Theorem. $\{x\} = \{x, x\}$.

---

[59] $f \subseteq x \times y \wedge (\forall u(u \in x) \to (\exists v(v \in y) \wedge (u, v) \in f \wedge (((u, v_1) \in f) \wedge (u, v_2) \in f) \to v_1 = v_2))$

In a given particular situation, the final theorem permits a synonymic language relation between the singleton and the pair class.

Conclusion

Analytical philosophy and *Algebra* have been the general framework of this analysis for discussing certain mathematical objects of class algebra and make certain logical-conceptual clarifications. The analysis made reference to a certain philosophical work, paradigmatic for analytical philosophy: Ludwig Wittgenstein's *Tractatus Logico-Philosophicus*. It has attempted to present a different, or at least complementary position regarding the relation of logic and mathematics, or more precisely of the propositions of logic and those of mathematics.

References

1. **Bernays Paul** - *Axiomatic Set Theory*, (Dover Books on Mathematics), Dover Publications, 1991
2. **Breaz Simion**, **Covaci Rodica** - *Elemente de logică, teoria mulțimilor și aritmetică*, Editura Fundației pentru Studii Europene, Cluj-Napoca, 2006
3. **Dauben Joseph** W. - *Georg Cantor and Pope Leo XIII: Mathematics, Theology and the Infinite*. In: Journal of the History of Ideas, 38/1, 85-108, 1977
4. **Ferreirós José** - *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics*, Birkhäuser Verlang AG, Basel - Boston - Berlin, 2007
5. **Lolli Gabriele** - *Filosofia della matematica. L'eredità del Novecento*, Società editrice il Mulino, Bologna, 2002
6. **Lionnais François** - *Le grands courants de la pensée mathématique*, Collection Histoire de la pensée, Hermann Éditeurs des Sciences et des Arts, Paris, 1998
7. **Mancosu Paolo** - *Visualization in logic and mathematics*. In: *Visualization, Explanation and Reasoning Styles in Mathematics* [Mancosu P., Jørgensen K. and Pedersen S. (Eds.)], Dordrecht: Springer, 2005
8. **Mancosu Paolo** - *Mathematical explanation: Why it matters?* In: *The philosophy of mathematical practice* [Mancosu P. (Ed.)], Oxford: Oxford University Press, 2008
9. **Muller F.A**. - *Cantor -Von Neumann Set Theory*, http://philsci-archive.pitt.edu/4603/1/CantorVonNeumann-2011.pdf
10. **Shapiro Stewart** - *Thinking About Mathematics*, Oxford: Oxford University Press, 2000
11. **Schindler Ralf-Dieter** - *Adilemma in the Philosophy of Set Theory*, Notre Dame Journal of Formal Logic, Volume 35, Number 3, Summer 1994 (https://wwwmath.uni-muenster.de/u/rds/dilemma.pdf)
12. **Suppes Patrick** - *Axiomatic Set Theory*, (Dover Books on Mathematics), Dover Publications, 1972

13. **Tappedent Jamie** - *Proof Style and Understanding in Mathematics: Visualization, Unification and Axiom Choice*. In *Visualization, Explanation and Reasoning Styles in Mathematics* [Mancosu P., Jørgensen K. and Pedersen S. (Eds.)], Dordrecht: Springer, 2005
14. **Thom René** - *Apologie du logos*, Hachette, Paris, 1990
15. **Wittgeinstein Ludwig** - *Cours sur les fondements des mathématiques-Cambridge 1939*, Éditions T.E.R., Mauvezin, 1995
16. **Wittgenstein Ludwig** - *Tractatus Logico-Philosophicus* (1922): Project Gutenberg, 2010 [EBook #5740], www.gutenberg.org